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ENERGY DECAY OF SOLUTIONS FOR THE VISCOELASTIC NONLINEAR EQUATION WITH A INTERNAL TIME-VARYING DELAY TERM AND BOUNDARIES RELATING THE KIRCHHOFF TYPE COEFFICIENT

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Abstract. In this work, we conduct research that includes the following boundaries related to the main equation which is the viscoelastic Kirchhoff type equation with the following nonlinear source and time-varying delay

$$u(x,t) = 0 \qquad \text{on} \quad \Gamma_0 \times [0,+\infty),$$

$$\left(M(x,t,\|\nabla u(t)\|^2) + \nu^2\right) \frac{\partial u(x,t)}{\partial n} + g(u_t) = 0 \quad \text{on} \quad \Gamma_1 \times [0,+\infty).$$

Under the smallness conditions from boundaries related to initial condition with respect to nonlinear coefficient and the relaxation function and other assumptions, we prove the energy decay rates of solutions for the Kirchhoff type energy.

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1. Introduction

This research focuses on the following problem:

$$(1)\ u_{tt}(x,t) - M(x,t,\|\nabla u(t)\|^2)\Delta u(x,t) + \int_0^t h(t-\tau)div[a(x)\nabla u(\tau)]d\tau$$
$$+|u|^\gamma u + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-s(t)) = 0 \quad \text{in} \quad \Omega\times\mathbb{R}^+,$$
$$(2)\ u_t(x,t-s) = z_0(x,t) \qquad \text{in} \quad \Omega\times[-s(0),0),$$
$$(3)\ u(x,t) = 0 \qquad \text{on} \quad \Gamma_0\times\mathbb{R}^+,$$
$$(4)\ \left(M(x,t,\|\nabla u(t)\|^2) + \nu^2\right)\frac{\partial u(x,t)}{\partial n} + g(u_t) = 0 \qquad \text{on} \quad \Gamma_1\times\mathbb{R}^+,$$
$$(5)\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad \text{in} \quad \Omega,$$

where Ω be a bounded open set of $\mathbb{R}^N (N \geq 1)$ with a piecewise-smooth boundary of class C^2. We consider Γ_0, Γ_1 a partition of Γ, with Γ_0 and Γ_1 having positive Lebesgue measures and $\overline{\Gamma_0}\cap\overline{\Gamma_1} = \emptyset$. Let n be the outward normal to Γ. γ is positive. Other conditions such as M, h, a be in next section. Moreover, μ_1 and μ_2 are real numbers in that μ_1 is only a positive constant, $s > 0$ represents the time-varying delay. In fact, u_0, u_1 z_0 are initially given functions belonging to suitable space and $u(x,t)$ is the transversal displacement of the strip at spatial coordinate x and time t in the real world application.

Time delays frequently occur in various physical, chemical, biological, thermal, and economic phenomena. Recently, the study of controlling PDEs with time delay effects has gained significant attention. For example, see [1, 2] and the references therein. The presence of a delay could be a stabilizing influence. Even a very small delay can destabilize a system, preventing issues like stick-slip in the mass production process for mechanical engineering.

This issue originates from the mathematical modeling of axially moving viscoelastic materials in real-world systems. Viscoelastic materials are renowned for their inherent damping properties, attributed to their unique ability to remember past deformations. Mathematically, these damping effects are represented using integro-differential operators. Moreover, the stability effects are influenced by time-varying delays. Consequently, our problem (1)-(5) does not include weak or strong damping terms. Our purpose, unlike the previous results [3] and [4], is focused not only on the memory effects but also on the time-varying delay and the boundaries considering nonlinear coefficients in the problem. Recently, numerous authors have explored issues related to Timoshenko or basic hyperbolic types for viscoelastic materials (see [5, 6]). Additionally, various researchers address the asymptotic behavior of solutions, nonlinear functions, and control theory considering viscoelasticity (see [7–10]). Moreover, numerous engineering devices experience transverse vibrations in axially moving strings. This model is widely utilized, particularly for long and narrow subjects with negligible flexural rigidity, such as threads, wires, magnetic tapes, belts, band saws, and cables. To better understand the linear or nonlinear dynamic behavior of these moving continua, various mathematical models and simulations have been developed [11–17]. The mathematical model for axially moving strings was first introduced by Kirchhoff [18] (and see Carrier [11]), and the original equation is presented as follows:

$$\rho h \frac{\partial^2 u}{\partial t^2} = \Big(p_0 \ + \ \frac{Eh}{2L}\int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big)\frac{\partial^2 u}{\partial x^2}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate x and time t; E, the young's modulus; ρ, the mass density; h, the cross section area; L, the length; and p_0, the initial axial tension.Recently, numerous authors have examined problems involving the extended Kirchhoff type equation, which pertains to axially moving heterogeneous or non-heterogeneous materials, including the nonlinear vibrations of beams, strings, plates, and membranes (see [5–7]).

In this paper, we will primarily focus on the decay rate of energy in a viscoelastic system with Kirchhoff-type boundaries, considering an internal time-varying delay term. We derive the proof by employing smallness condition functions related to the Kirchhoff coefficient, the relaxation function, and internal time-varying delay. Essentially, the energy difference comprises Kirchhoff-type potential energy and internal time-varying delay.

The main focus of this paper is the treatment of the boundaries in the estimation of energy. The principal idea of this paper is that the coefficient of the norm derived through calculations is negative due to the assumption of the smallness of $M(\cdot)$, which can minimize the impact on the boundedness of energy.

This paper is structured as follows: In Section 2, we introduce the necessary notations and materials (including assumptions, lemmas, etc.) for our study and present a theorem on global existence and energy decay rate (the main result). Section 3 provides the proof of our main result.

2. Preliminaries and main results

Initially, we present the fundamental bracket pairing within $\Omega \subset \mathbb{R}^N$.

$$\langle \varphi, \psi \rangle \equiv \int_\Omega (\varphi, \psi) dx,$$

provided that $(\varphi, \psi) \in L^1(\Omega)$. And we set the norms as follows.

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$

For the sake of simplicity, we denote $\|u\|_{L^2(\Omega)}, \|u\|_{L^1(0,+\infty)}, \|v\|_{L^\infty(0,+\infty)}$ by $\|u\|, \|v\|_{L^1}, \|v\|_{L^\infty}$ respectively.

Throughout this paper, we define

$$V = \{u | u \in H^1(\Omega), u = 0 \text{ on } \Gamma_0\},$$

In the following sections, we outline the general hypotheses.

(A$_1$) $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function satisfying $h(0) > 0$, and there exists positive constant $t_0, \zeta_1, \zeta_2, \zeta_3$ such that

$$-\zeta_1 \le h'(t) \le -\zeta_2 h(t), \quad \forall t > t_0,$$
$$0 \le h''(t) \le \zeta_3 h(t), \quad \forall t > t_0.$$

(A$_2$) $a : \Omega \to \mathbb{R}^+$ is a nonnegative bounded function and $a(x) \ge a_0 > 0$ on Ω with

$$\frac{m_0}{a_0} \ge 1 - \|a\|_\infty \int_0^\infty h(s) ds = l > 0,$$

where m_0 is in (B$_2$). And also, the following smallness condition satisfy

$$\epsilon_7 < a_0^2 \int_0^t h(s) ds.$$

(A$_3$) γ satisfies

$$0 \le \gamma \le \frac{2}{n-2}, \quad n \ge 3,$$
$$\gamma \ge 0, \quad n = 1, 2.$$

(A$_4$) The initial data satisfy

$$u_0 \in V \cap H^2(\Omega), \quad u_1 \in V,$$
$$\frac{\partial u_0}{\partial n} + g(u_1) = 0 \text{ on } \Gamma_1.$$

(B$_1$) $M(x, t, \lambda)$ is a real-valued function of class C^2 on $x \in \overline{\Omega},\ t \ge 0,\ \lambda \le 0$.

(B$_2$) $0 < m_0 \le M(x,t,\lambda) \le C_0 f(\lambda)$ with $M(x,t,\lambda) = M_1(x,t) + M_2(x,t,\lambda)$. And also, the following smallness condition satisfy

$$f(\lambda) < \sqrt{\frac{\frac{a_0 h(t)}{2} - C_p\widetilde{C_1} + \epsilon_2\left(m_0 - \frac{1}{2}\right)}{\epsilon_3\epsilon_8}}.$$

(B$_3$) $\frac{\partial M_1}{\partial t} \le 0$, $\left|\frac{\partial M_2}{\partial t}\right| \le C_1 g_1(\lambda)$, $\left|\frac{\partial M}{\partial \lambda}\right| \le C_2 g_2(\lambda)$, $0 < m_1 \le M_x(x,t,\lambda)$.

(B$_4$) $f, g_1, g_2 \in C^1([0,+\infty); \mathbb{R}_+)$ are strictly increasing. Furthermore, C_i $(i = 0,1,2)$ is a positive constant.

(C$_1$) There exists a non-increasing differential function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\zeta(t) > 0, h'(t) \le -\zeta(t)h(t) = 0, \quad \forall t > 0.$$

(C$_2$) The function g is a nondecreasing C^1 function and $g(0) = 0$. Furthermore, there exist positive constants α and β such that

$$g(\xi)\xi \ge \alpha|\xi|^2, \qquad \forall \xi \in \mathbb{R},$$

$$|g(\xi)| \le \beta|\xi|, \qquad \forall \xi \in \mathbb{R}.$$

For the time-varying delay, we assume as in [1] that there exist positive constants $s_0, \overline{s}$ such that

$$0 < s_0 \le s(t) \le \overline{s}, \quad \forall t > 0. \tag{6}$$

Moreover, we assume that the speed of the delay satisfies

$$s'(t) \le d < 1, \quad \forall t > 0, \tag{7}$$

which is

$$s \in W^{2,\infty}([0,T]), \quad \forall t > 0$$

and that μ_1, μ_2 satisfy

$$|\mu_2| < \sqrt{1-d}\,\mu_1. \tag{8}$$

As in [1], let us introduce the function

$$z(x,\varrho,t) = u_t(x, t - s(t)\varrho), \quad x \in \Omega,\ \varrho \in (0,1),\ t > 0.$$

Then, the problem (1)-(5) is equivalent to

$$
\begin{aligned}
&(9)\ u_{tt}(x,t) - M(x,t,\|\nabla u(t)\|^2)\Delta u(x,t) + \int_0^t h(t-\tau)div[a(x)\nabla u(\tau)]d\tau \\
&\qquad +|u|^\gamma u + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) = 0 \quad \text{in} \quad \Omega\times(0,+\infty),\\
&(10)\ s(t)z_t(x,\varrho,t) + (1-s'(t)\varrho)z_\varrho(x,\varrho,t) \quad \text{in} \quad \Omega\times(0,1)\times(0,+\infty),\\
&(11)\ u_t(x,t) = z(x,0,t) \quad \text{on} \quad \Omega\times(0,+\infty),\\
&(12)\ z(x,\varrho,t) = z_0(x,-\varrho s(0)) \quad \text{in} \quad \Omega\times(0,1),\\
&(13)\ u(x,t) = 0 \quad \text{on} \quad \Gamma_0\times[0,+\infty),\\
&(14)\ \big(M(x,t,\|\nabla u(t)\|^2) + \nu^2\big)\frac{\partial u(x,t)}{\partial n} + g(u_t) = 0 \quad \text{on} \quad \Gamma_1\times[0,+\infty),\\
&(15)\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega,
\end{aligned}
$$

In the following, we give a lemma which will be useful in this paper.

LEMMA 2.1. *Denote* $(h\diamond u)(t) = \int_0^t h(t-\tau)\|\sqrt{a(x)}(u(t)-u(\tau))\|^2 d\tau$. *Then we have*

(16)

$$
\begin{aligned}
\int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau = &-\frac{1}{2}\frac{d}{dt}\left[(h\diamond u)(t)\right] + \frac{1}{2}(h'\diamond u)(t)\\
&+\frac{1}{2}\frac{d}{dt}\left[\|\sqrt{a(x)}\nabla u(t)\|^2\int_0^t h(s)ds\right]\\
&-\frac{1}{2}h(t)\|\sqrt{a(x)}\nabla u(t)\|^2.
\end{aligned}
$$

Proof. A direct computation shows that

$$\begin{aligned}
\int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau =& \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau) - a(x)\nabla u(t), \nabla u'(t)\rangle d\tau \\
&+ \int_0^t h(t-\tau)\langle a(x)\nabla u(t), \nabla u'(t)\rangle d\tau \\
=& -\frac{1}{2}\int_0^t h(t-\tau)\left[\frac{d}{dt}\|\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))\|^2\right] d\tau \\
&+ \frac{1}{2}\int_0^t h(t-\tau)\left[\frac{d}{dt}\|\sqrt{a(x)}\nabla u(t)\|^2\right] d\tau \\
=& -\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(t-\tau)\|\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))\|^2 d\tau\right] \\
&+ \frac{1}{2}\int_0^t h'(t-\tau)\|\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))\|^2 d\tau \\
&+ \frac{1}{2}\frac{d}{dt}\int_0^t h(t-\tau)\|\sqrt{a(x)}\nabla u(t)\|^2 d\tau \\
&- \frac{1}{2}h(t)\|\sqrt{a(x)}\nabla u(t)\|^2.
\end{aligned}$$

□

LEMMA 2.2. **(General Poincaré Inequality).**
Denote $H^1_{\Gamma_0}(\Omega) = \{u | u \in H^1(\Omega), u|_{\Gamma_0} = 0\}$ *and* $meas(\Gamma_0) > 0$. *Then there exists a positive constant* B *such that* $\|u\|_{L^2(\Omega)} \leq B\|\nabla u\|_{L^2(\Omega)}$, *for all* $u \in H^1_{\Gamma_0}(\Omega)$.

Proof. The proof can be found in [22]. □

Then, we can state our result as follows.

THEOREM 2.3. *Let the assumptions* $(A_1), (A_4)$, (B_1)-(B_4) *and* (C_1) *hold. Then there exists a unique solution* u *of the problem* (9)-(15) *satisfying*

$$u \in L^\infty(0,T; V \cap H^2(\Omega)),\ u' \in L^\infty(0,T;V),\ u'' \in L^\infty(0,T;L^2(\Omega)),$$

and

$$u(x,t) \to u_0(x) \text{ in } V \cap H^2(\Omega); \qquad u'(x,t) \to u_1(x) \text{ in } V;$$

moreover,

$$z(x,\varrho,t) \to z_0(x) \text{ in } L^2(\Omega \times (0,1)),$$
$$u' \in L^\infty(0,T; L^2(\Gamma_1),$$

as $t \to 0$.

Proof. By using Galerkin's approximation and a routine procedure similar to that of cite [5, 20], by using Lemma 2.1 and Lemma 2.2, we can the global existence result for the solution subject to (1)-(5) under the assumptions (A_1)-(A_4), (B_1)-(B_4) and (C_1)-(C_1). □

THEOREM 2.4. *Let u be the global solution of the problem* (1)-(5) *with the above all conditions. We define the Kirchhoff type energy functional $E(t)$ as*

$$E(t) = \frac{1}{2}\left[\|u'(t)\|^2 + \int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx + \frac{2}{\gamma+2}\|u'(t)\|_{\gamma+2}^{\gamma+2}\right] + \frac{\zeta}{2}\int_{t-s(t)}^{t}\int_\Omega e^{\eta(s-t)}u_t^2(s)dxds,$$

where $\zeta,\ \eta$ are suitable positive constants.
Then the energy functional decays exponentially to zero as the time goes to infinity, that is,

$$E(t) \le \kappa e^{-\vartheta t},\ \forall t \ge 0$$

where κ, ϑ are positive constants.

3. Proof of Theorem 2.4 (Energy decay)

Proof. Multiplying u' on both sides of Eq.(1), integrating the resulting equations over Ω, and using the Green formula and (3), we have

$$\begin{aligned}(17)\qquad \langle u''(t), u'(t)\rangle &+ \langle M(x,t,\|\nabla u(t)\|^2)\nabla u(t), \nabla u'(t)\rangle \\ &+ \langle M_x(x,t,\|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\ &+ \left\langle \nu^2\frac{\partial u(t)}{\partial n} + g(u_t), u'(t)\right\rangle_{\Gamma_1} \\ &- \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau + \langle |u|^\gamma u, u'\rangle \\ &+ \langle \mu_1 u_t(x,t) + \mu_2 u_t(x,t-s(t)), u'\rangle = 0,\end{aligned}$$

that is

(18)

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}M_1(x,t)|\nabla u(x,t)|^2dx \\
&+ \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}M_2(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2dx \\
&+ \left[\int_\Omega \frac{\partial}{\partial \lambda}M_2(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2dx\right]\left\langle \nabla u'(t), \nabla u(t)\right\rangle \\
&- \langle M_x(x,t,\|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\
&- \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau \\
&+ \frac{\zeta}{2}\int_\Omega u_t^2(t)dx - \frac{\zeta}{2}\int_\Omega e^{-\eta s(t)}u_t^2(t-s(t))(t-s'(t))dx \\
&- \left\langle \nu^2\frac{\partial u(t)}{\partial n} + g(u_t), u'(t)\right\rangle_{\Gamma_1} \\
&- \frac{\eta\zeta}{2}\int_{t-s(t)}^t\int_\Omega e^{-\eta(s-t)}u_t^2(s)dxds,
\end{aligned}$$

where

(19)

$$\begin{aligned}
E(t) &= \frac{1}{2}\left[\|u'(t)\|^2 + \int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2dx + \frac{1}{\gamma+2}\|u'(t)\|_{\gamma+2}^{\gamma+2}\right] \\
&+ \frac{\zeta}{2}\int_{t-s(t)}^t\int_\Omega e^{\eta(s-t)}u_t^2(s)dxds.
\end{aligned}$$

From (B_3) and Hölder inequality, and (6), (7) and some mainipulations as in [1], we obtain

(20)
$$
\begin{aligned}
E'(t) \le{} & \|u(t)\|^2 \left\{ \frac{C_1}{2} g_1(\|\nabla u(t)\|^2) + C_2 g_2(\|\nabla u(t)\|^2) \|\nabla u'(t)\| \|u(t)\| \right\} \\
& - \langle M_x(x,t,\|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\
& - \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau \\
& - \left(\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\zeta}{2}\right) \int_\Omega u_t^2(t)dx \\
& - \left(e^{-\eta\bar{s}}\frac{\zeta(1-d)}{2} - \frac{|\mu_2|\sqrt{1-d}}{2}\right) \int_\Omega u_t^2(t-s(t))dx \\
& - \left\langle \nu^2 \frac{\partial u(t)}{\partial n} + g(u_t), u'(t) \right\rangle_{\Gamma_1} \\
& - \frac{\eta\zeta}{2} \int_{t-s(t)}^t \int_\Omega e^{-\eta(s-t)} u_t^2(s) dx ds.
\end{aligned}
$$

By (B_3), (16) and Young's inequality, we have

$$
\begin{aligned}
E'(t) \le{} & \widetilde{C_1}\|u(t)\|^2 + \epsilon_1 m_1 \|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 \\
& - \frac{1}{2}\frac{d}{dt}\left[(h \diamond u)(t)\right] + \frac{1}{2}(h' \diamond \nabla u)(t) \\
& + \frac{1}{2}\frac{d}{dt}\left[\|\sqrt{a(x)}\nabla u(t)\|^2 \int_0^t h(s)ds\right] \\
& - \frac{1}{2}h(t)\|\sqrt{a(x)}\nabla u(t)\|^2 \\
& - \left(\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\zeta}{2}\right) \int_\Omega u_t^2(t)dx \\
& - \left(e^{-\eta\bar{s}}\frac{\zeta(1-d)}{2} - \frac{|\mu_2|\sqrt{1-d}}{2}\right) \int_\Omega u_t^2(t-s(t))dx \\
& - \left(\alpha - \frac{\beta\nu^2}{m_0+\nu^2}\right)\|u'(t)\|_{\Gamma_1}^2 - m_3\|u'(t)\|^2 \\
& - \frac{\eta\zeta}{2} \int_{t-s(t)}^t \int_\Omega e^{-\eta(s-t)} u_t^2(s) dx ds,
\end{aligned} \tag{21}
$$

where

$$\widetilde{C_1} = \frac{C_1}{2} g_1(\|\nabla u(t)\|^2) + C_2 g_2(\|\nabla u(t)\|^2)\|\nabla u'(t)\|\|u(t)\| \tag{22}$$

is a positive constant. And ϵ_1 is also a positive constant.
Define the new energy functional $E_1(t)$ as follows

$$E_1(t) = E(t) + \frac{1}{2}(h \diamond \nabla u)(t) - \frac{1}{2}\|\sqrt{a(x)}\nabla u(t)\|^2 \int_0^t h(s)ds. \tag{23}$$

For positive constants ϵ_2 and ϵ_3, let us define the perturbed modified energy by

$$F(t) = E_1(t) + \epsilon_2 \varphi(t) + \epsilon_3 \psi(t), \tag{24}$$

where

$$\varphi(t) = \langle u'(t), u(t) \rangle. \tag{25}$$

and

$$\psi(t) = -\int_0^t h(t-\tau)\langle a(x)u'(t), u(t) - u(\tau)\rangle d\tau. \tag{26}$$

By using the Cauchy's inequality, Hölder inequality and Poincarè inequality, there exist positive constants α_1, α_2 such that for each $t > 0$

$$\alpha_1 F(t) \leq E_1(t) \leq \alpha_2 F(t). \tag{27}$$

Proposition 3.1. **(Energy equivalence)**

$$\alpha_1 F(t) \leq E_1(t) \leq \alpha_2 F(t) \quad \textit{for all } t \geq 0,$$

where α_1 and α_2 are positive constants.

Proof. Now, we will fix ζ in the energy $E(t)$ such that

$$2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}} - \zeta > 0, \tag{28}$$

$$\zeta - \frac{|\mu_2|}{\sqrt{1-d}} > 0 \tag{29}$$

and

$$\eta < \frac{1}{s}\left|\log \frac{|\mu_2|}{\zeta\sqrt{1-d}}\right|. \tag{30}$$

Then, similar as Proposition 3.1. in [3], we can choose two constants α_1 and α_2. In fact, the existence of such a constant η is guaranteed by the assumption (8). □

Using the trace theorem and Cauchy's inequality, and noting (C_2), we obtain

(31)
$$|\langle g(u')u\rangle_{\Gamma_1}| \le \|g(u')\|_{\Gamma_1}\|u\|_{\Gamma_1} \le \beta\lambda\|u'\|_{\Gamma_1}\|\nabla u\| \le \epsilon_3\|\nabla u\|^2 + \frac{\beta^2\lambda^2}{4\epsilon_3}\|u'\|_{\Gamma_1}^2$$

Then from (A_1) and (21),(23) and (28)-(31), we have

(32)
$$\begin{aligned} E_1'(t) \le& \|u(t)\|^2\widetilde{C_1} + \epsilon_1 m_1\|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 \\ & - \frac{\zeta_2}{2}(h\diamond\nabla u)(t) - \frac{1}{2}a_0h(t)\|\nabla u(t)\|^2 \\ & - C_2\int_\Omega [u_t^2(t) + u_t^2(t-s(t))]dx \\ & - \frac{\beta^2\lambda^2}{4\epsilon_3}\left(1 - \frac{\nu^2}{m_0+\nu^2}\right)\|\nabla u(t)\|_{\Gamma_1}^2 \\ & - \frac{\eta\zeta}{2}\int_{t-s(t)}^t\int_\Omega e^{-\eta(s-t)}u_t^2(s)dxds \\ \le& \|u(t)\|^2\widetilde{C_1} + \epsilon_1 m_1\|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 \\ & - \frac{\beta^2\lambda^2}{4\epsilon_3}\left(1 - \frac{\nu^2}{m_0+\nu^2}\right)\|\nabla u(t)\|_{\Gamma_1}^2 \\ & - \frac{\zeta_2}{2}(h\diamond\nabla u)(t) - \frac{1}{2}a_0h(t)\|\nabla u(t)\|^2 - C_2\int_\Omega u_t^2(t-s(t))dx, \end{aligned}$$

where, C_2 is some positive constant. Since m_0 is positive due to (B_2), we no longer need to consider the norm on the boundary. And also, by (A_2), the energy $E_1(t)$ is a positive functional. Applying Poincarè inequality to (32), we deduce

$$\begin{aligned} E_1'(t) \le& \left(C_p\widetilde{C_1} + \epsilon_1 m_1 - \frac{1}{2}a_0h(t)\right)\|\nabla u(t)\|^2 \\ & + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 - \frac{\zeta_2}{2}(h\diamond\nabla u)(t) - C_2\int_\Omega u_t^2(t-s(t))dx, \end{aligned} \tag{33}$$

where C_p is the Poincarè coefficient. Meanwhile, we note from (A_1) and (A_2) that

(34)

$$\begin{aligned}E_1(t) \geq& \frac{1}{2}\|u(t)\|^2 + \frac{1}{2}\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx\\ &+ \frac{1}{2}\left(1 - \|a\|_\infty \int_0^t h(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(h \diamond u)(t)\\ &+ \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2} + \frac{\zeta}{2}\int_{t-s(t)}^t \int_\Omega e^{\eta(s-t)} u_t^2(s) dx ds\\ \geq& l\Big[\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2}\\ &+ \frac{\zeta}{2}\int_{t-s(t)}^t \int_\Omega e^{\eta(s-t)} u_t^2(s) dx ds\Big].\end{aligned}$$

So, we deduce the relation $0 \leq E(t) \leq l^{-1}E_1(t)$. Therefore, the uniform decay of $E(t)$ is a result of the decay of $E_1(t)$. In fact, using (1), we have

$$\begin{aligned}\varphi'(t) =& \langle u''(t), u(t)\rangle + \|u'(t)\|^2.\\ =& \|u'(t)\|^2 + \Big\langle u(t), M(x,t,\|\nabla u(t)\|^2)\Delta u(x,t)\\ &- \int_0^t h(t-\tau) div[a(x)\nabla u(\tau)] d\tau - |u(t)|^\gamma u(t)\\ &- \mu_1 u_t(x,t) - \mu_2 u_t(x,t-s(t))\Big\rangle\\ =& \|u'(t)\|^2 - \int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(t)|^2 dx\\ &+ \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u(t)\rangle] d\tau - |u(t)|^\gamma u(t)\\ &- \mu_1 \int_\Omega u(t)u_t(t) dx - \mu_2 \int_\Omega u(t) u_t(t-s(t)) dx.\end{aligned} \tag{35}$$

By Cauchy inequality and Young's inequality, we have

(36)

$$\begin{aligned}
&\left|\int_0^t h(t-\tau)\left\langle a(x)\nabla u(\tau),\nabla u(t)\right\rangle] d\tau\right| \\
\le&\frac{1}{2}\|\nabla u(t)\|^2+\frac{1}{2}\left\|\int_0^t h(t-\tau)(a(x)|\nabla u(\tau)-\nabla u(t)|+a(x)|\nabla u(t)|)d\tau\right\|^2 \\
\le&\frac{1}{2}\|\nabla u(t)\|^2+\left(\frac{1}{2}+\frac{1}{8\epsilon_6}\right)\left\|\int_0^t h(t-\tau)a(x)|\nabla u(\tau)-\nabla u(t)|d\tau\right\|^2 \\
&+\left(\frac{1}{2}+\frac{\epsilon_6}{2}\right)\left\|\int_0^t h(t-\tau)a(x)|\nabla u(t)|d\tau\right\|^2,
\end{aligned}$$

where ϵ_6 with respect to Young's inequality is a positive constant. Using the assumption (A_2) and (36), we get

(37)

$$\begin{aligned}
&\left|\int_0^t h(t-\tau)\left\langle a(x)\nabla u(\tau),\nabla u(t)\right\rangle] d\tau\right| \\
\le&\left(\frac{1}{2}+\frac{1}{8\epsilon_6}\right)\|a\|_\infty\int_0^t h(s)ds\int_0^t h(t-\tau)\left\|\sqrt{a(x)}(\nabla u(\tau)-\nabla u(t))\right\|^2 d\tau \\
&+\left(\frac{1}{2}+\frac{\epsilon_6}{2}\right)\|\nabla u(t)\|^2\left(\|a\|_\infty\int_0^t h(s)a(x)ds\right)^2+\frac{1}{2}\|\nabla u(t)\|^2 \\
\le&\frac{1}{2}(1+(1+\epsilon_6)(1-l)^2)\|\nabla u(t)\|^2+\frac{(4\epsilon_6+1)(1-l)}{8\epsilon_6}(h\diamond\nabla u)(t).
\end{aligned}$$

Also, using Young's and Poincaré's inequalities gives

$$-\mu_1\int_\Omega u(t)u_t(t)dx\le\varepsilon\int_\Omega|\nabla u|^2dx+C(\varepsilon)\int_\Omega u_t^2(t)dx \tag{38}$$

$$-\mu_2\int_\Omega u(t)u_t(t-s(t))dx\le\varepsilon\int_\Omega|\nabla u|^2dx+C(\varepsilon)\int_\Omega u_t^2(t-s(t))dx \tag{39}$$

By combining (35) and (37)-(39), we conclude

(40)

$$\begin{aligned}
\varphi'(t)\le&(1+C(\varepsilon))\|u'(t)\|^2+\frac{1}{2}(1-2m_0+(1+\epsilon_6)(1-l)^2+2\varepsilon)\|\nabla u(t)\|^2 \\
&+\frac{(4\epsilon_6+1)(1-l)}{8\epsilon_6}(h\diamond\nabla u)(t)-\|u(t)\|_{\gamma+2}^{\gamma+2} \\
&+C(\varepsilon)\int_\Omega u_t^2(t-s(t))dx.
\end{aligned}$$

Next, we estimate $\psi'(t)$ as follows. In fact, using (1), we have

(41)

$$
\begin{aligned}
\psi'(t) =& -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t)-u(\tau)\rangle d\tau.\\
&-\int_0^t h(t-\tau)\langle a(x)u''(t), u(t)-u(\tau)\rangle d\tau - \|\sqrt{a(x)}u'(t)\|^2\int_0^t h(s)ds\\
=& -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t)-u(\tau)\rangle d\tau.\\
&-\int_0^t h(t-\tau)\langle M(x,t,\|\nabla u(t)\|^2)a(x)\nabla u(t), \nabla u(t)-\nabla u(\tau)\rangle d\tau\\
&-\left\langle \int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau, \int_0^t h(t-\tau)a(x)(\nabla u(t)-\nabla u(\tau))d\tau\right\rangle\\
&+\int_0^t h(t-\tau)\langle a(x)|u|^\gamma u, u(t)-u(\tau)\rangle d\tau\\
&-\|\sqrt{a(x)}u'(t)\|^2\int_0^t h(s)ds\\
&+\int_\Omega\left(\int_0^t h(t-\tau)a(x)(u(t)-u(\tau))ds\right)[\mu_1 u_t(t)+\mu_2 u_t(t-s(t))]dx.
\end{aligned}
$$

Using Cauchy inequality, Poincarè inequality and (A_1), we have

$$
\begin{aligned}
&\left|-\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t)-u(\tau)\rangle d\tau\right|\\
\leq&\epsilon_7\|\nabla u(t)\|^2+\frac{\zeta_1}{4\epsilon_7}\left\|\int_0^t h(t-\tau)a(x)|u(t)-u(\tau)|d\tau\right\|^2 \qquad (42)\\
\leq&\epsilon_7\|\nabla u(t)\|^2+\frac{\zeta_1}{4\epsilon_7}(1-l)C_p^2(h\diamond\nabla u)(t),
\end{aligned}
$$

where ϵ_7 is a positive constant with respect to Cauchy inequality and C_p is the Poincarè coefficient. Similarly, using Cauchy inequality and (B_2), we get

(43)

$$
\begin{aligned}
&\left|-\int_0^t h(t-\tau)\langle M(x,t,\|\nabla u(t)\|^2)a(x)\nabla u(t), \nabla u(t)-\nabla u(\tau)\rangle d\tau\right|\\
\leq&\epsilon_8 f^2(\|\nabla u(t)\|^2)\|u'(t)\|^2+\frac{C_0(1-l)}{4\epsilon_8}(h\diamond\nabla u)(t)
\end{aligned}
$$

and

(44)

$$\begin{aligned}
&\left|-\left\langle\int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau, \int_0^t h(t-\tau)a(x)(\nabla u(t)-\nabla u(\tau))d\tau\right\rangle\right| \\
\le&\epsilon_9\left\|\int_0^t h(t-\tau)(a(x)|\nabla u(t)-\nabla u(\tau)|+a(x)|\nabla u(t)|)d\tau\right\|^2 \\
&+\frac{1}{4\epsilon_9}\left(\|a\|_\infty\int_0^t h(s)ds\right)\int_0^t h(t-\tau)\|\sqrt{a(x)}(\nabla u(t)-\nabla u(\tau))\|^2d\tau \\
\le&2\epsilon_9\left(\left\|\int_0^t h(t-\tau)a(x)|\nabla u(t)-\nabla u(\tau)|d\tau\right\|^2+\left\|\int_0^t h(t-\tau)a(x)|\nabla u(t)|d\tau\right\|^2\right) \\
&+\frac{1-l}{4\epsilon_9}(h\diamond\nabla u)(t) \\
\le&\left(2\epsilon_9+\frac{1}{4\epsilon_9}\right)(1-l)(h\diamond\nabla u)(t)+2\epsilon_9(1-l)^2\|\nabla u(t)\|^2,
\end{aligned}$$

where ϵ_8, ϵ_9 are positive constants with respect to Cauchy inequality. And also, using Cauchy inequality and Poincarè inequality, we have

$$\begin{aligned}
&\left|\int_0^t h(t-\tau)\langle a(x)|u(t)|^\gamma u, u(t)-u(\tau)\rangle d\tau\right| \\
\le&\epsilon_{10}\|u(t)\|_{2(\gamma+1)}^{2(\gamma+1)}+\frac{C_p(1-l)}{4\epsilon_{10}}(h\diamond\nabla u)(t),
\end{aligned} \tag{45}$$

where ϵ_{10} is a positive constant with respect to Cauchy inequality and C_p is the Poincarè coefficient. Noting $H^1(\Omega)\hookrightarrow L^{2(\gamma+1)}(\Omega)$ and using Poincarè inequality, (23), (32) and (45), we get

$$\begin{aligned}
&\left|\int_0^t h(t-\tau)\langle a(x)|u(t)|^\gamma u, u(t)-u(\tau)\rangle d\tau\right| \\
\le&\epsilon_{10}C_p^{2(\gamma+1)}\left(\frac{2E_1(0)}{l}\right)^\gamma\|\nabla u(t)\|^2+\frac{C_p(1-l)}{4\epsilon_{10}}(h\diamond\nabla u)(t),
\end{aligned} \tag{46}$$

where C_p is the Poincarè coefficient. And also, we get

(47)

$$\begin{aligned}
&\left|\int_\Omega\left(\int_0^t h(t-\tau)a(x)(u(t)-u(\tau))ds\right)[\mu_1u_t(t)+\mu_2u_t(t-s(t))]dx\right| \\
\le&\epsilon_{10}\int_\Omega[u_t^2(t)+u_t^2(t-s(t))]dx+\frac{C_p(1-l)}{4\epsilon_{10}}(h\diamond\nabla u)(t),
\end{aligned}$$

Combining (37)-(44) and (46)-(47) and also using (A_2), we deduce

(48)

$$\begin{aligned}\psi'(t) \le & \left(\epsilon_7 - a_0^2\int_0^t h(s)ds + \epsilon_{10}\right)\|u'(t)\|^2\\ & + \left(\epsilon_8 f^2(\|\nabla u(t)\|^2) + 2\epsilon_9(1-l)^2 + \epsilon_{10}C_p^{2(\gamma+1)}\left(\frac{2E_1(0)}{l}\right)^\gamma\right)\|\nabla u(t)\|^2\\ & + \left(\frac{\zeta_1}{4\epsilon_7}C_p^2 + \frac{C_0}{4\epsilon_8} + 2\epsilon_9 + \frac{1}{4\epsilon_9} + \frac{C_p}{4\epsilon_{10}}\right)(1-l)(h\diamond\nabla u)(t)\\ & + \epsilon_{10}\int_\Omega u_t^2(t-s(t))dx.\end{aligned}$$

Combining (33), (24), (40) and (48), we deduce

(49)

$$\begin{aligned} F'(t) =& E_1'(t) + \epsilon_2\varphi'(t) + \epsilon_3\psi'(t)\\ \le & w_1\|u'(t)\|^2 + w_2\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2dx + w_3(h\diamond\nabla u(t))\\ & - \|u(t)\|_{\gamma+2}^{\gamma+2} + w_4\int_\Omega u_t^2(t-s(t))dx,\end{aligned}$$

where

$$w_1 = \frac{m_1}{4\epsilon_1} + (1+C(\varepsilon))\epsilon_2 + \epsilon_3\left(\epsilon_7 - a_0^2\int_0^t h(s)ds + \epsilon_{10}\right),$$

$$\begin{aligned} w_2 =& f(\|\nabla u(t)\|^2)C_0\left[C_p\widetilde{C_1} + \epsilon_1 m_1 - \frac{1}{2}a_0h(t)\right]\\ & + \frac{\epsilon_2 f(\|\nabla u(t)\|^2)C_0}{2}(1 - 2m_0 + (1+\epsilon_6)(1-l)^2 + 2\varepsilon)\\ & + \epsilon_3 f(\|\nabla u(t)\|^2)C_0\left(\epsilon_8 f^2(\|\nabla u(t)\|^2) + 2\epsilon_9(1-l)^2 + \epsilon_{10}C_p^{2(\gamma+1)}\left(\frac{2E_1(0)}{l}\right)^\gamma\right),\end{aligned}$$

$$w_3 = -\frac{\zeta_2}{2} + \left[\frac{\epsilon_2(4\epsilon_6+1)}{8\epsilon_6} + \epsilon_3\left(\frac{\zeta_1}{4\epsilon_7}C_p^2 + \frac{C_0}{4\epsilon_8} + 2\epsilon_9 + \frac{1}{4\epsilon_9} + \frac{C_p}{4\epsilon_{10}}\right)\right](1-l),$$

$$w_4 = \epsilon_2 C(\varepsilon) + \epsilon_3\epsilon_{10} - C_2$$

By using the smallness condition in (A_2) and (B_2), for the fixed $\epsilon_i, i = 1, 4, \cdots, 10$, we choose $\epsilon_j > 0, j = 2, 3$ and ε small enough such that $w_k < 0, k = 1, 2, 3, 4$. According to (23) and (49), there exist a positive constant s such that

$$F(t) \le -sE_1(t) \tag{50}$$

for all t which is larger than the fixed time T_0. We conclude from (27) and (50) that

$$F(t) \leq -s\alpha_1 F(t)$$

for all t which is larger than the fixed time T_0. That is, for all t which is larger than the fixed time T_0,

$$F(t) \leq F(T_0)e^{s\alpha_1 T_0}e^{-s\alpha_1 t}. \tag{51}$$

Therefore, we deduce from (27), (34) and (51) that there are positive constants κ and ϑ such that

$$E(t) \leq \kappa \exp\{-\vartheta t\} \quad for\ all\ t \geq 0\ and\ as\ t \to +\infty.$$

□

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